

Smooth concordance of links topologically concordant to the Hopf link

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Abstract. It was shown by Jim Davis that a 2-component link with Alexander polynomial one is topologically concordant to the Hopf link. In this paper, we show that there is a 2-component link with Alexander polynomial one that has unknotted components and is not smoothly concordant to the Hopf link, answering a question of Jim Davis. We construct infinitely many concordance classes of such links, and show that they have the stronger property of not being smoothly concordant to the Hopf link with knots tied in the components.

1. Introduction

The study of odd-dimensional link concordance has complications that go beyond the study of knot concordance of the individual components. In this paper, we discuss some additional differences that arise in the classical dimension when one also considers the distinction between smooth and topological concordance. We consider the question of whether a 2-component link is smoothly concordant to the Hopf link. A well-known theorem of M. Freedman [11, 12] states that a knot whose Alexander polynomial is one is topologically concordant to the trivial knot. Completing a program initiated by J. A. Hillman [16], J. Davis [10] showed that a 2-component link with (multivariable) Alexander polynomial one is topologically concordant to the Hopf link.

It follows directly from the existence of smoothly non-slice knots with Alexander polynomial one that Davis’ theorem cannot hold in the smooth category. A “more refined” question in this setting was posed by Davis: is there a 2-component link with Alexander polynomial one which is not smoothly concordant to the Hopf link, but each of whose components is smoothly concordant to the unknot? In Section 3, we provide such a link; in fact we prove a stronger result, whose statement benefits from a bit of terminology.

Given a link $L = (L_1, \dots, L_n)$, and a split link $L' = (L'_1, \dots, L'_n)$, one can form the connected sum $L \# L' = (L_1 \# L'_1, \dots, L_n \# L'_n)$. Since L' is split, the connected sum is well-defined. We will refer to the resulting link as a *locally knotted* L . Let H denote the Hopf link and L' a 2-component link. If both components of L' are knots of trivial Alexander polynomial, then the locally knotted Hopf link $H \# L'$ will have (multivariable) Alexander polynomial equal to one. Note that if one component of L' is not smoothly slice, then $H \# L'$ is certainly not smoothly concordant to H ; this is the point of Davis' question.

Theorem A. *There is a 2-component link with Alexander polynomial one which has unknotted components and is not smoothly concordant to any locally knotted Hopf link.*

In Section 3, we give two proofs for Theorem A which use the Ozsváth-Szabó τ invariant [21]. In Theorem 4.1 using the d -invariant (or correction term [20]) we will in fact show that there are infinitely many smooth concordance classes of such links.

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2. Covering links and blow-down

All links will be assumed to be oriented. Generally speaking, concordance will refer to smooth concordance, with the adjective ‘topological’ (always meaning topologically locally flat) added as appropriate. Links are always ordered. We will generally use the same letters for a link and its components, so that for example L_1 and L_2 would indicate the first and second components of a 2-component link L .

We offer two related proofs of Theorem A. The first uses the technique of covering links [6] while the second comes from observations on blow-down for links.

Covering link calculus

In this paper we will use the following construction, which is a special case of covering link calculus formulated more generally in [6] for links with arbitrary number of components in \mathbb{Z}_p -homology spheres (see also [8, 5, 7]).

Let p be a prime. For a link $L = L_1 \cup L_2$ in S^3 , consider the p^a -fold cyclic cover of S^3 branched along L_2 , say Y , and then consider a component, say J , of the pre-image of L_1 . Viewing J as a knot in Y , we call J a *covering knot* of L . Though the construction and the lemma below apply to more general cases, in this paper we will always apply these to a 2-component link L in S^3 with L_2 unknotted and $\text{lk}(L_1, L_2) = 1$, so that Y is S^3 again and J is the whole pre-image of L_1 .

The following is a well-known fact, which holds in both topological and smooth category.

Lemma 2.1 (e.g., see [8, 5, 6]). *Suppose L and L' are concordant links. Then their p^a -fold covering knots J and J' are rationally concordant in the sense of [4, 2], i.e., concordant in a rational homology $S^3 \times I$.*

Corollary 2.2. *Suppose L is concordant to the Hopf link. Then the p^a -fold covering knot of L is rationally slice, i.e. bounds a 2-disk in a rational homology 4-ball.*

Proof. A covering knot of the Hopf link is obviously unknotted. From Lemma 2.1, the conclusion follows immediately. \square

Blow-down for links

Recall that the result of ± 1 surgery on an unknot in S^3 is again S^3 . If L is a link with an unknotted component L_n , then doing ± 1 surgery on L_n produces a new link \check{L} in S^3 . We say that \check{L} is obtained by blowing down L_n (with framing specified as necessary).

Lemma 2.3. *Suppose L and L' are concordant, and that L_n and L'_n are both unknots. Then \check{L} and \check{L}' are concordant in a homotopy $S^3 \times I$.*

The lemma holds in either the topological or smooth category; in the topological category we can of course replace the homotopy $S^3 \times I$ by the real one.

Proof. Let $C = (C_1, C_2, \dots)$ be the concordance in $S^3 \times I$. Following Gordon's classic paper [14] we can do the ± 1 surgery on the component C_n , to produce a simply-connected homology cobordism between S^3 and itself. Since the surgery took place in the complement of $\bigcup_{i \neq n} C_i$, those components give a concordance \check{C} . \square

We will say that the concordance \check{C} is obtained by blowing down C_n . In the special case when $L' = H$, the Hopf link, note that \check{L}'_1 is the unknot. Hence we obtain an immediate corollary.

Corollary 2.4. *Let L be a 2-component link such that the component L_2 is unknotted. If L is concordant to the Hopf link, then the knot \check{L}_1 obtained by blowing down L_2 is homotopically slice, that is, slice in a homotopy 4-ball.*

Local knotting

For a link $L = (L_1, \dots, L_n)$ let $S(L)$ denote the split link with the same components as L , thus $S(L) = L_1 \amalg \dots \amalg L_n$. We will make use of the following observation regarding components of concordant links.

Lemma 2.5. *Suppose that L and J are links with the same number of components and that J has unknotted components. If L is concordant to \tilde{J} , a locally knotted J , then L is concordant to $J \# S(L)$. In particular, if L and J have unknotted components and L is concordant to a locally knotted J , then L is concordant to J .*

Proof. Let C be a concordance from L to $\tilde{J} = J \# L'$ for some split link L' . Denote by \tilde{C}_j the concordance from the component L'_j of L' to the corresponding component L_j of L obtained by turning C_j upside down. Take the product concordance from \tilde{J} to itself, and sum each component $L'_j \times I$ with a copy of \tilde{C}_j , to obtain a concordance from \tilde{J} to $J \# S(L)$. Composing C with this concordance gives a concordance from L to $J \# S(L)$. \square

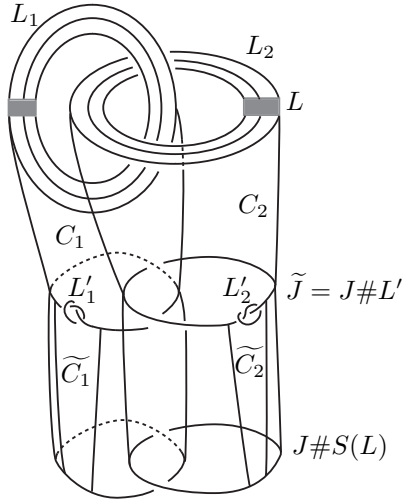
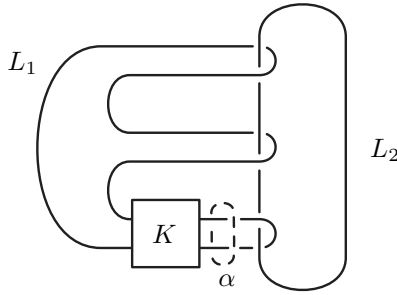


FIGURE 1. Schematic illustration of proof of Lemma 2.5

3. Examples and proof of Theorem A

Consider the link $L = L(K)$ pictured in Figure 2, where for the moment K is an arbitrary knot in S^3 (cf. [3, Figure 1]). The notation means that the band labelled K should be tied in the knot K , in such a way that the framing of the band is 0. The dotted curve α is not a component of the link, but is used in the description of $L(K)$ as an ‘infection’.

FIGURE 2. The link $L(K)$

Alexander polynomial of $L(K)$

Proposition 3.1. *The link $L(K)$ has the following properties:*

- (1) *Both components L_1 and L_2 are unknotted.*
- (2) *The multivariable Alexander polynomial $\Delta_{L(K)}(x, y) = 1$.*

Proof. (1) is obvious. Observe that if K were the unknot, then $L(K)$ is just the Hopf link. Also, $L(K)$ is obtained from the Hopf link by removing a tubular neighborhood of α and filling it in with the exterior of K ; in the resulting 3-manifold, which is S^3 , the Hopf link becomes $L(K)$. It is well known that this operation preserves the homology of

the universal cover of the link exterior (and consequently the Alexander polynomial) if α is null-homologous in the link complement. For example, to show this one may apply Mayer-Vietoris and uses that a knot exterior is a homology $S^1 \times D^2$. It follows that $L(K)$ has Alexander polynomial one. \square

We remark that to prove (2) above, one may use the method of D. Cooper [9], which computes the Alexander polynomial from matrices defined from Seifert surfaces for the two components having only clasp-type intersections. In our case, one can use the surfaces S_1 and S_2 in Figure 3. The recipe in [9] is to take a basis for $H_1(S_1 \cup S_2)$, and then derive

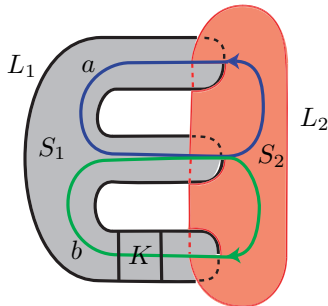


FIGURE 3. Surfaces for Cooper's method

a Seifert-type matrix recording linking numbers amongst suitable pushoffs of the curves in this basis. In our case, the curves a and b in Figure 3 form such a basis. Since the linking numbers of the relevant pushoffs of a and b are independent of the choice of K , it follows that the Alexander polynomial of $L(K)$ is equal to that of $L(\text{unknot})$ which is the Hopf link. Namely, $\Delta_{L(K)} = 1$ for any K .

Proof of Theorem A

We present two proofs of Theorem A based on the Ozsváth-Szabó τ invariant [21], using the topological mechanisms described in Section 2. In the following section, we will give a third proof, using the d -invariant (or correction term [20]) that yields a stronger result (stated as Theorem 4.1).

First proof of Theorem A. Let $L = L(K)$ be the link illustrated in Figure 2 where K is a knot with positive τ -invariant. For instance, K could be chosen to be the right-handed trefoil. Denote the components of L by L_2 and L_1 as in Figure 2. We consider the covering knot J of L obtained by taking the double cover of S^3 branched along the component L_2 ; J is the pre-image of L_1 in the resulting S^3 . A standard cut-paste argument along the obvious 2-disk bounded by L_2 shows that J has the knot type of the positive Whitehead double $\text{Wh}(K \# K^r)$, where K^r denotes the orientation reverse of K .

From the hypothesis we have $\tau(K \# K^r) = 2\tau(K) > 0$. Therefore by a result of Manolescu-Owens [19], $J = \text{Wh}(K \# K^r)$ is not rationally slice. (See also Hedden [15], which gives $\tau(J) = 1$.) Consequently L is not concordant to the Hopf link, by Corollary 2.2.

Since the components of L are unknotted, and L is not concordant to the Hopf link, Lemma 2.5 implies that it is in fact not concordant to a locally knotted Hopf link. \square

Second proof of Theorem A. We will show that $L = L(K)$ is not concordant to the Hopf link, making use of a recent τ -invariant calculation by Adam Levine [17]. He considers a

generalized Whitehead double $D_{J,s}(K,t)$, defined (roughly) as a plumbing of two annuli, tied into knots J and K , with s and t twists respectively. The case $s = -1$ and $J = \mathcal{O}$ (the unknot) corresponds to the t -twisted positive Whitehead double of K . The knot in Figure 4 is the knot $D_{\mathcal{O},-2}(K,0)$, and Levine [17, Proposition 2.5] computes its τ -invariant to be

$$(1) \quad D_{\mathcal{O},-2}(K,0) = \begin{cases} 0 & \text{if } \tau(K) \leq 0, \\ 1 & \text{if } \tau(K) > 0. \end{cases}$$

Again we choose K to be a knot with $\tau(K) > 0$ for L . Let \check{L}_1 be the knot in S^3 obtained from L by blowing down L_2 with positive framing. One can easily see that \check{L}_1 is the knot $D_{\mathcal{O},-2}(K,0)$. If L were concordant to the Hopf link, then as in Corollary 2.4, the knot \check{L}_1 would be homotopically slice. However, by Levine's calculation, $\tau(\check{L}_1) = 1$, which means that \check{L}_1 is not homotopically slice. \square

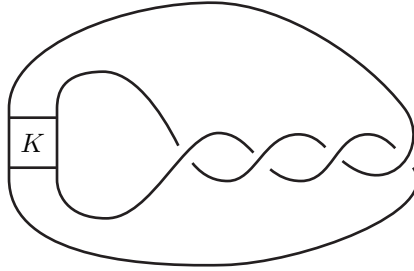


FIGURE 4. $L(K)$ after blowing down L_2

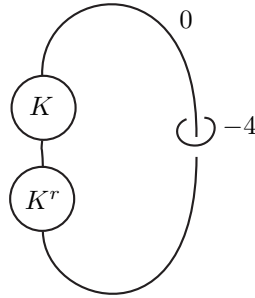
A minor variation on the second proof may be obtained using the work of Rudolph [23], coupled with the observation of Livingston [18] that the results of [23] apply as well to surfaces lying in a homology ball, rather than B^4 . The knot \check{L}_1 pictured in Figure 4 may be described as the boundary of the plumbing of two annuli. One consists of two parallel copies of the knot K , with linking number 0, and the other is an unknotted annulus that twists -2 times about its core. As in the proof of [23, Lemma 2], if K is strongly quasi-positive, then the knot \check{L}_1 is also strongly quasi-positive. So if we choose K to be non-trivial and strongly quasi-positive, then \check{L}_1 is not homotopically slice, and hence $L(K)$ is not concordant to the Hopf link.

4. Infinitely many concordance classes

By varying the choice of K , we can obtain infinitely many examples of links that have the properties stated in Theorem A. The verification that these links are not concordant to one another gives an alternate proof of Theorem A.

Theorem 4.1. *There is a sequence of knots $K(n)$ ($n \in \mathbb{N}$) such that the links $L(K(n))$ satisfy the conclusion of Theorem A and are distinct up to smooth concordance.*

Proof. For any knot K , consider the knot $\check{L}_1(K)$ obtained by blowing down the second component of $L(K)$; there is an evident genus 1 Seifert surface for $\check{L}_1(K)$ visible in Figure 4. Using the algorithm of Akbulut and Kirby [1], it is then easy to draw a surgery picture for $M_2(\check{L}_1(K))$, the double cover of S^3 branched over $\check{L}_1(K)$, illustrated below. By the

FIGURE 5. Double cover of $(\check{L}_1(K))$

slam-dunk move [13] this is diffeomorphic to $S^3_{1/4}(K \# K^r)$, a homology 3-sphere.

Let $K(n)$ be the $(2, 2n + 1)$ torus knot, and write $L(n)$ for $L(K(n))$. Suppose that $L(m)$ and $L(n)$ are concordant. Blow down the concordance between $L_2(m)$ and $L_2(n)$ to get a concordance in a homotopy $S^3 \times I$ between $\check{L}_1(m)$ and $\check{L}_1(n)$. Now, we take the 2-fold branched cover over that concordance, to get a \mathbb{Z}_2 -homology cobordism between $M_2(\check{L}_1(m))$ and $M_2(\check{L}_1(n))$. This implies that the d -invariants (or correction term [20]) of these manifolds, in the trivial Spin^c structure, are equal.

The torus knots are reversible, and alternating so that d -invariants of their surgeries may be computed from their Alexander polynomials and signatures. In particular, from [20, Corollary 9.14] and [22, Corollary 1.5] we get

$$d(S^3_{1/4}(K(n) \# K(n)^r)) = d(S^3_1(K(n) \# K(n)^r)) = -2n.$$

It follows that for positive $m \neq n$, the links $L(m)$ and $L(n)$ are not concordant and also not concordant to the Hopf link. Again by Lemma 2.5, $L(n)$ with $n > 0$ is not concordant to any locally knotted Hopf link. \square

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